

Automorphic Black Holes as Probes of Extra Dimensions

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Abstract:

Recent progress in the understanding of the statistical nature of black hole entropy shows that the counting functions in certain classes of models are determined by automorphic forms of higher rank. In this paper we combine these results with Langlands' reciprocity conjecture to view black holes as probes of the geometry of spacetime. This point of view can be applied in any framework leading to automorphic forms, independently of the degree of supersymmetry of the models. In the present work we focus on the class of Chaudhuri-Hockney-Lykken compactifications defined as quotients associated to \mathbb{Z}_N groups. We show that the black hole entropy of these CHL_N models can be derived from elliptic motives, thereby providing the simplest possible geometric building blocks of the Siegel type entropy count.

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1 Introduction

Black holes can be viewed as objects that encode structural information about the theories in which they are embedded. As such they provide probes that can be used to ask what exactly we could learn about the ambient physical theory if we were able to perform experiments with them. One particular focus in black hole physics over the past four decades has been the problem of a fundamental understanding of their entropy, in particular its statistical interpretation. The purpose of this paper is to address the question what kind of information is encoded in the automorphic entropy functions that have recently been constructed for certain types of black holes. The idea developed here is to view black hole entropy as a probe that is sensitive to the geometric structure of the extra dimensions predicted by string theory.

The microscopic understanding of black hole entropy has made great progress in the past few years. In the context of $\mathcal{N} = 4$ compactifications these results have lead to partition functions that provide a Boltzmann count of the numbers of states. A class of models that has received much attention is based on Chaudhuri-Hockney-Lykken constructions obtained via quotients with respect to cyclic groups \mathbb{Z}_N , denoted here by CHL_N [1]. Dijkgraaf-Verlinde-Verlinde [2], Jatkar-Sen [3], and Govindarajan-Krishna [4] have shown that the entropy of certain types of dyonic black holes is completely determined by Siegel modular forms Φ^N of genus two, the structure of which depends on the quotient group \mathbb{Z}_N of the CHL_1 compactification manifold T^6 (see also ref. [5]). Siegel modular forms define a special class of automorphic forms that generalize to the symplectic groups $\text{Sp}(2n, \mathbb{Z})$ the classical modular forms derived from congruence subgroups of the modular group $\text{Sp}(2, \mathbb{Z}) \cong \text{SL}(2, \mathbb{Z})$. It has been known for more than a century that certain modular forms have a geometric origin, and generalizations to automorphic forms have been discussed more recently. Such experimental results have led to a web of conjectures by Taniyama, Shimura, Weil, and later Langlands and others, that raise the hope that at least certain classes of automorphic forms are of geometric origin. In string theory the notion of geometric automorphic forms has previously been used to provide a construction of the compactified geometry directly in terms of modular forms on the worldsheet, leading to a framework that realizes the idea of an emergent spacetime in string theory [6].

The geometric construction of automorphic forms and their associated representations provides meaning to the general question whether it is possible to deduce an underlying irreducible geometric structure that leads to the automorphic forms that appear in black hole entropy counting problems in string theory, and if so, whether these geometric structures are unique.

In this paper we address this problem in the context of the CHL_N theories and their associated Siegel modular forms. Not much has been proven about the geometric interpretation of Siegel modular forms even in the special case of genus $n = 2$, but we will see that the conjectural framework of Langlands applied to Siegel forms implies that it is not possible to find geometric structures in the CHL_N models that directly support the black hole Siegel forms in the form usually envisioned. For this reason it is useful to first analyze the precise structure of the CHL_N type Siegel forms Φ^N in more detail before addressing the question of the geometric origin of these objects, and thus of the entropy of the CHL_N black holes. The first simplification that arises in the context of the CHL_N models is that the Siegel forms Φ^N encoding the entropy of the CHL_N type black holes are lifts of simpler types of modular forms. The lifts relevant for the CHL_N black holes were first considered by Maaß and Skoruppa for modular forms of level one, and later generalized to higher level. It was shown that the Siegel forms Φ^N of the CHL_N model belong to the so-called Maaß Spezialschar, and are determined by classical modular cusp forms $f^N \in S_w(\Gamma_0(N))$ of level N with respect to the Hecke group $\Gamma_0(N)$ of some weight w . These classical forms are Hecke eigenforms that are determined by the electric (or magnetic) BPS states, and lead to Φ^N by composing two maps, the Skoruppa lift from classical forms to Jacobi forms, and the Maaß lift from Jacobi forms to Siegel modular forms. These lifts are completely canonical, independent of N , hence the forms f^N provide the key building blocks of the dyonic black hole count. In the following the classical modular forms f^N will be called the Maaß-Skoruppa roots, or the black hole roots, of the Siegel modular forms Φ^N .

The problem of a geometric understanding of the black hole entropy in CHL_N models therefore translates into the problem of understanding the geometric origin of the Maaß-Skoruppa roots. It is this question that we address in this paper. We show that while the motives naively associated to the Maaß-Skoruppa roots are not physical, it is possible to reduce these modular

forms further, and to construct all the forms f^N of the class of CHL_N models in terms of classical modular forms $f_2^{\tilde{N}}$ of weight two, where the level \tilde{N} is determined by N . Our lift construction therefore implies that motivically these forms are supported by elliptic curves $E_{\tilde{N}}$ of conductor \tilde{N} , which are determined up to isogeny. The view of black holes as probes of the geometry of spacetime raises the question in what detail the entropy probes the extra dimensions. The fact that the dyon counts of the CHL_N models are determined by elliptic curves $E_{\tilde{N}}$ shows that the underlying geometry of the entropy is that given by a single motive, not by the composite motivic structure expected from more complicated compactification manifolds. In general higher dimensional manifold are determined by several modular, or automorphic, motives. Hence the fact that for each model there is a single modular form shows that the CHL_N black hole entropy functions considered so far do not probe the full structure of the compact geometry.

The outline of the paper is as follows. In Section 2 we briefly describe the microscopic structure of CHL_N black holes, in particular the lift structure of the Siegel modular forms that is relevant for this paper. In Section 3 we describe the general motivic framework associated to Siegel modular forms, and in Section 4 we derive the underlying motivic building blocks of the Siegel forms that arise in the CHL_N models. In Section 5 we summarize our work, and in an Appendix we analyze the symmetry structure of the forms that appear in our discussion.

2 Black hole entropy of the CHL_N models

2.1 CHL_N models

The first step toward a generalization of $\mathcal{N} = 4$ black hole entropy was taken by Jatkar and Sen [3]. These authors formulated a proposal for the dyonic partition functions in the class of Chaudhuri-Hockney-Lykken models CHL_N for N prime. The compact manifolds in these models are quotient spaces with respect to some cyclic group $\mathbb{Z}_N := \mathbb{Z}/N\mathbb{Z}$ of order N

$$\text{CHL}_N : \quad \text{Het}(T^6/\mathbb{Z}_N) \cong \text{IIA}((K3 \times T^2)/\mathbb{Z}_N). \quad (1)$$

In the IIA frame the group \mathbb{Z}_N acts via a symplectic automorphism on the K3 factor and as an order N shift on one of the 1-cycles of the torus T^2 .

The reason why Siegel modular forms arise in the CHL_N models can be traced to the duality group, which for the low energy supergravity theory is given by

$$U_N(\mathbb{R}) = \text{SL}(2, \mathbb{R}) \times \text{SO}(6, r_N - 6, \mathbb{R}), \quad (2)$$

where r_N is the rank of the gauge group of the CHL_N theory. In the full string theory of the CHL_N models this group is broken to the subgroup [7]

$$U_N(\mathbb{Z}) = \Gamma_1(N) \times \text{SO}(6, r_N - 6, \mathbb{Z}), \quad (3)$$

where

$$\Gamma_1(N) = \left\{ \gamma \in \text{SL}(2, \mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}. \quad (4)$$

Invariance under the duality group implies that the dyon degeneracies, a priori functions of the charges

$$\text{CHL}_N : (Q_e, Q_m) \in \mathbb{L}_N \oplus \mathbb{L}_N, \quad (5)$$

where $\mathbb{L}_N = \mathbb{L}^{6, r_N - 6}$ is a Narain lattice, depend only on the duality invariant norms, given by

$$(Q_e^2, Q_m^2, Q_e Q_m) \in \frac{2}{N} \mathbb{Z} \times 2\mathbb{Z} \times \mathbb{Z}. \quad (6)$$

Physical quantities invariant under T-duality then should depend on the charges through their three invariant norms.

2.2 Siegel automorphic forms of genus 2

The fact that there are three T-invariant norms suggests to introduce three chemical potentials, associated to $Q_m^2, Q_e^2, Q_e Q_m$, denoted here by (τ, σ, ρ) . The partition function of the CHL_N models are then expected to be expressed in terms of a 3-variable automorphic forms $\Phi(\tau, \sigma, \rho)$ as

$$Z(\tau, \sigma, \rho) = \frac{1}{\Phi(\tau, \sigma, \rho)} = \sum_{k, \ell, m} d(k, \ell, m) q^k r^\ell s^m, \quad (7)$$

where $q = e^{2\pi i\tau}$, $r = e^{2\pi i\sigma}$, $s = e^{2\pi i\rho}$, and the Fourier expansion of the form can be written as

$$\Phi(q, r, s) = \sum_{k, \ell, m} g(k, \ell, m) q^k r^\ell s^m, \quad (8)$$

with k, ℓ, m are integers determined by the T-duality invariant norms in eq. (6). The detailed structure of the form Φ will depend on the structure of the models considered.

A well-known class of automorphic forms are Siegel modular forms of genus n , defined as functions

$$\Phi_w : \mathcal{H}_n \longrightarrow \mathbb{C} \quad (9)$$

on the Siegel upper halfplane

$$\mathcal{H}_n = \left\{ T \in M_n(\mathbb{C}) \mid T \text{ symmetric, with positive - definite imaginary part} \right\} \quad (10)$$

of dimension $\dim_{\mathbb{C}} \mathcal{H}_n = \frac{n}{2}(n+1)$. Siegel modular forms of genus 2 are therefore defined on a three-dimensional space, and it is natural to check whether such Siegel forms provide a useful framework for CHL_N models.

Like classical modular forms, Siegel modular forms are characterized by a weight w and a level N , determined by the relevant congruence subgroup $\Gamma^{(n)}(N)$ of the symplectic group $\text{Sp}(2n, \mathbb{Z})$. The functions Φ_w satisfy a scaling behavior with respect to elements $M \in \Gamma^{(n)}(N) \subset \text{Sp}(2n, \mathbb{R})$, where the action of M on \mathcal{H}_n is defined as

$$MT = \begin{pmatrix} A & B \\ C & D \end{pmatrix} T = (AT + B)(CT + D)^{-1}. \quad (11)$$

In the following analysis only scalar Siegel modular forms are needed, for which the transformation behavior is given by

$$\Phi_w(MT) = j_w(M, T) \Phi_w(T) = \det(CT + D)^w \Phi_w(T), \quad (12)$$

where w is assumed to be integral.

The full symplectic group $\text{Sp}(2n, \mathbb{Z})$ is too large to allow many Siegel forms at fixed weight, and it is necessary to consider congruence subgroups of the full symplectic group. This is similar to the case of classical modular forms, where the full modular group $\text{SL}(2, \mathbb{Z})$ is too

restrictive as well to allow for many interesting modular forms at fixed weight. As in the classical case there are different types of congruence groups that are of interest for different questions. In the context of CHL_N models the groups of interest are $\Gamma_0^{(n)}(N) \subset \text{Sp}(2n, \mathbb{Z})$, defined for arbitrary genus n as

$$\Gamma_0^{(n)}(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2n, \mathbb{Z}) \mid C \equiv 0 \pmod{N} \right\}. \quad (13)$$

These groups generalize to $\text{Sp}(4, \mathbb{Z})$ the Hecke congruence subgroup $\Gamma_0(N) \subset \text{SL}(2, \mathbb{Z})$ of the full modular group $\text{SL}(2, \mathbb{Z})$.

The Fourier expansion of $\Phi_w(T)$ takes the form

$$\Phi_w(T) = \sum_{\substack{0 \leq U^t = U \\ \text{semi-integral}}} g(U) e^{2\pi i \text{tr}(UT)}, \quad (14)$$

where semi-integral means that the diagonal entries of U are integers, while the off-diagonal entries are either integers or half-integers. For $n = 2$ the variables of the Siegel upper plane take the form

$$T = \begin{pmatrix} \tau & \rho \\ \rho & \sigma \end{pmatrix} \in \mathcal{H}_2 \quad (15)$$

with $\tau, \sigma \in \mathcal{H}_1$, i.e.

$$\text{Im}(\tau) > 0, \quad \text{Im}(\sigma) > 0, \quad \text{Im}(\tau)\text{Im}(\sigma) > \text{Im}(\rho)^2. \quad (16)$$

The functional dependence is often written as $\Phi_w(\tau, \sigma, \rho) = \Phi_w(T)$, and the Fourier expansion can be expressed via

$$U = \begin{pmatrix} k & m/2 \\ m/2 & \ell \end{pmatrix}, \quad \text{with } k, \ell, m \in \mathbb{Z}, \quad k, \ell \geq 1, \quad m^2 < 4k\ell,$$

as

$$\Phi_w(T) = \sum_{\substack{k, \ell \in \mathbb{N}, m \in \mathbb{Z} \\ k, \ell, 4k\ell - m^2 > 0}} g(k, \ell, m) q^k r^\ell s^m. \quad (17)$$

The Fourier coefficients $g(k, \ell, m)$ determine the degeneracies $d(k, \ell, m)$ via the partition function (7).

2.3 The Maaß-Skoruppa lift for CHL_N Siegel modular forms

We have seen that the duality invariance of the CHL_N model suggests to look for three-dimensional automorphic forms, leading to Siegel modular forms of genus two as the simplest candidates. As noted above, Siegel forms are characterized like classical modular forms by their weight $w^N = w(\Phi^N)$ and their level. It turns out that their weight is given in terms of the rank r_N of the gauge group of the CHL_N model as

$$w^N = \frac{1}{2}(r_N - 8). \quad (18)$$

We will identify the Siegel forms Φ^N by their level instead of the weight. The rank r_N of the \mathbb{Z}_N -model depends not only on the order of the quotient group \mathbb{Z}_N but also on the precise form of the action. For the models considered here they are given by in Table 1.

| N | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-------|----|----|----|----|----|----|----|----|
| r_N | 28 | 20 | 16 | 14 | 12 | 12 | 10 | 10 |

Table 1. Ranks r_N of the CHL_N models.

The embedding of the chemical potentials into the genus two Siegel upper halfplane \mathcal{H}_2 shows that in the limit $\rho \rightarrow 0$ the Siegel form $\Phi^N(\tau, \sigma, \rho)$ should factorize as

$$\Phi^N(\tau, \sigma, \rho) \xrightarrow{\rho \rightarrow 0} \sim \alpha(\rho) f^N(\tau) g^N(\sigma), \quad (19)$$

where $f^N(\tau)$ corresponds to purely electrically charged states, while $g^N(\sigma)$ corresponds to purely magnetically charged states. Electro-magnetic duality leads to $f^N = g^N$.

The factorization (19) along the diagonal suggests that the Siegel modular forms describing the CHL_N models can be constructed as lifts of classical modular forms f^N . Such lifts have been constructed by Maaß [8] and Skoruppa [9] for the full modular group, and extensions for congruence groups have been discussed in refs. [10, 11]. We will call this construction the Maaß-Skoruppa lift, or additive lift. It is obtained via a two-step construction, the Skoruppa lift $\text{Sk}(f^N)$ from classical modular forms f^N to Jacobi forms φ^N , and the Maaß lift $\text{M}(\varphi^N)$ from Jacobi forms φ^N to Siegel forms Φ^N

$$f^N \xrightarrow{\text{Sk}} \varphi^N \xrightarrow{\text{M}} \Phi^N = \text{MS}(f^N). \quad (20)$$

We briefly outline these two lifts because this induction of Siegel forms by their Maaß-Skoruppa roots will be the starting point for our motivic interpretation described in the next section.

2.3.1 The Skoruppa lift

The first step of the additive lift from classical modular forms to CHL_N black hole Siegel forms is based on a result first shown by Skoruppa [9] for level one forms, and later extended by Cléry and Gritsenko [11] to modular forms of higher level. This construction implements a map from classical cusp forms to Jacobi forms.

Jacobi modular forms of weight w and index ℓ are maps

$$\varphi_{w,\ell} : \mathcal{H} \times \mathbb{C} \longrightarrow \mathbb{C} \quad (21)$$

such that for any element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in some congruence subgroup $\gamma \in \Gamma \subset \text{SL}(2, \mathbb{Z})$

$$\varphi_{w,\ell} \left(\frac{a\tau + b}{c\tau + d}, \frac{\rho}{c\tau + d} \right) = (c\tau + d)^w e^{2\pi i \ell c \frac{\rho^2}{(c\tau + d)}} \varphi_{w,\ell}(\tau, \rho). \quad (22)$$

Furthermore there is a transformation of the group \mathbb{Z}^2 , acting and transforming like

$$\varphi_{w,\ell}(\tau, \rho + \alpha\tau + \underline{\rho}) = e^{-2\pi i \ell \alpha(\alpha\tau + 2\rho)} \varphi_{w,\ell}(\tau, \rho), \quad (\alpha, \underline{\rho}) \in \mathbb{Z}. \quad (23)$$

Jacobi cusp forms admit a Fourier expansion as

$$\varphi_{w,\ell}(\tau, \sigma) = \sum_{k \in \mathbb{Z}_{\geq 0}} \sum_{\substack{m \in \mathbb{Z} \\ 4k\ell - m^2 > 0}} c(k, m) q^k s^m \quad (24)$$

while for general Jacobi forms the expansion is restricted by $4k\ell - m^2 \geq 0$. The space of Jacobi forms of weight w and index ℓ with respect to some congruence group $\Gamma \subset \text{SL}(2, \mathbb{Z})$ will be denoted by $J_{w,m}(\Gamma)$, or simply $J_{w,m}$.

The map sending cusp forms $f \in S_w(\Gamma_0(N), \epsilon)$ of weight w , level N , and character ϵ to Jacobi forms

$$\text{Sk} : S_{w+2}(\Gamma_0(N), \epsilon) \longrightarrow J_{w,1} \quad (25)$$

will be called the Skoruppa map. It is defined by multiplication with the prime form

$$K(\tau, \rho) = \frac{\vartheta_1(\tau, \rho)}{\eta^3(\tau)}, \quad (26)$$

given in terms of the Dedekind eta function

$$\eta(q) = \prod_{n \geq 1} (1 - q^n), \quad (27)$$

and the theta series $\vartheta_1(\tau, \sigma)$ defined as

$$\vartheta_1(q, s) = \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{1}{8}(2n+1)^2} s^{n+\frac{1}{2}}. \quad (28)$$

The lift is then given by

$$\varphi_{w,1}(\tau, \rho) := K^2(\tau, \rho) f(\tau). \quad (29)$$

The square of the prime form is one of the generators of the space of weak Jacobi forms of even weight and integral index $K^2(\tau, \rho) = \varphi_{-2,1}(\tau, \rho)$.

2.3.2 The Maaß-Skoruppa lift

The second step of the additive lift construction uses a result shown first by Maaß for level one modular forms, and later extended by Manickam, Ramakrishnan and Vesudan [10] to higher level. This Maaß lift constructs the Fourier coefficients $g(k, \ell, m)$ of the expansion (8) of the Siegel modular forms $\Phi_w(q, r, s)$ in terms of the coefficients $c(k, m)$ of the Fourier expansion (24) of a Jacobi form of weight w and index 1. For a Jacobi form of weight w and index ℓ the coefficients $c(k, m)$ depend on k, m only via the combination $(4k\ell - m^2)$. The Fourier coefficients of Siegel forms in the Maaß subspace are then given as

$$g(k, \ell, m) = \sum_{d|(k, \ell, m)} \chi(d) d^{w-1} c\left(\frac{k\ell}{d^2}, \frac{m}{d}\right), \quad (30)$$

where χ is a character that is either trivial or given by a Dirichlet character.

A more conceptual formulation of the Maaß lift can be obtained by noting that the Hecke operators T_m acting on Jacobi forms $\varphi_{w,\ell}(q, s)$ of weight w and index ℓ produce Jacobi forms $\varphi_{w,\ell+m}$ of the same weight and index $\ell + m$

$$T_m \varphi_{w,\ell} = \varphi_{w,\ell+m}. \quad (31)$$

With these operators one can then generate the Fourier-Jacobi expansion of the Siegel modular form as

$$\Phi_w(q, r, s) = \sum_{\ell} \varphi_{w,\ell}(q, s) r^{\ell} = \sum_{\ell} (T_{\ell} \varphi_{w,1})(q, s) r^{\ell}. \quad (32)$$

The Maaß-Skoruppa lift, obtained by combining the Maaß lift with the Skoruppa lift $M \circ \text{Sk}$, therefore maps classical modular forms to Siegel forms of genus 2

$$\Phi_w = M(\varphi_{w,1}) = M(\text{Sk}(f_{w+2})) =: \text{MS}(f_{w+2}). \quad (33)$$

In the context of the CHL_N models we will characterize the modular forms by their model index N rather than the weight. The weight $(w+2)$ of the forms $f_{w+2} = f^N$ is determined by the order N of the CHL_N group, as described in the next subsection.

2.4 A simple choice for the electric (magnetic) modular forms

The present subsection describes a rationale that identifies unique candidates for the CHL_N model Maaß-Skoruppa roots in a very simple way. First, recall that the S-duality of electromagnetism generalizes to the toroidal model as $\text{SL}(2, \mathbb{Z})$. It turns out that in the $N = 1$ model considered in [2] the electric modular form $f^1(q)$ is in fact the unique eta product with the respect to the full modular group.

Recall next that the duality group $\text{SL}(2, \mathbb{Z})$ of the $N = 1$ model is broken for higher N to the congruence group $\Gamma_1(N) \subset \text{SL}(2, \mathbb{Z})$ [7]. Assuming that the generalization of the electric BPS counting form $f^N(q)$ for arbitrary $N > 1$ generalizes in the simplest possible way the $N = 1$ partition function $f^1(q) = \eta(q)^{24}$ leads to a simple guess: it is natural to expect that the generalization $f^N(q)$ of $f^1(q)$ is given by modular forms of level N . For each prime order $N = p \in \{2, 3, 5, 7\}$ there is a unique candidate cusp form η -product with the appropriate level and integral weight. These forms are given can be written in closed form in terms of the Dedekind eta function as

$$f^N(q) = \eta(q)^{w+2} \eta(q^N)^{w+2} \in S_{w+2}(\Gamma_0(N), \epsilon_N), \quad (34)$$

where for $N = 1, 2, 3, 5, 7, 11$ the resulting weight is given by $w + 2 = \frac{24}{(N+1)}$, and $\Gamma_0(N) \subset \text{SL}(2, \mathbb{Z})$ is Hecke's congruence group. The character ϵ_N is only nontrivial for level $N = 7$, in which case it is given by the Legendre character $\epsilon_7(d) = \chi_{-7}(d) = \left(\frac{-7}{d}\right)$. In general, the Legendre character is defined as

$$\chi_N(p) = \left(\frac{N}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv N \pmod{p} \text{ is solvable} \\ -1 & \text{if } x^2 \equiv N \pmod{p} \text{ is not solvable} \\ 0 & \text{if } p|N. \end{cases} \quad (35)$$

For the composite values $N = 4, 6, 8$ that complete the CHL_N sequence of models the quotient $24/(N+1)$ is neither integral nor half-integral. It is natural to extend the above sequence for prime order by considering forms of weight

$$w + 2 := \left\lceil \frac{24}{N+1} \right\rceil, \quad (36)$$

where $\lceil a \rceil$ denotes the next largest integral number obtained from the rational number a . For $N = 4, 6, 8$ these forms therefore are of weight 5, 4, 3, respectively. Extending furthermore the expectation that the order of the group again determines the level of the modular form leads to unique candidates of eta products given by

$$\begin{aligned} f^4(\tau) &= \eta(\tau)^4 \eta(2\tau)^2 \eta(4\tau)^4 \in S_5(\Gamma_0(4), \chi_{-1}) \\ f^6(\tau) &= (\eta(\tau) \eta(2\tau) \eta(3\tau) \eta(6\tau))^2 \in S_4(\Gamma_0(6)) \\ f^8(\tau) &= \eta(\tau)^2 \eta(2\tau) \eta(4\tau) \eta(8\tau)^2 \in S_3(\Gamma_0(8), \chi_{-2}). \end{aligned} \quad (37)$$

The characters are again given by Legendre characters. The forms obtained in eqs. (34) and (37) are precisely the forms proposed by Jatkar and Sen [3] for prime orders, and by Govindarajan and Krishna [4] for the composite orders.

Using as input for the Skoruppa lift the forms $f^N \in S_{w+2}(\Gamma_0(N), \epsilon_N)$, with weights $(w+2)$ given by (36), leads to Jacobi forms $\varphi^N(q, s)$ of weight w and index 1. The Maaß lift of $\varphi^N(q, s)$ then leads to Siegel modular forms $\Phi^N(q, r, s) \in S_w(\Gamma_0^{(2)}(N))$.

The final step of the Siegel formulation of the microscopic entropy is motivated by the map between the diagonal and dominant zero divisors of the Siegel form. This map is obtained by

an $\text{Sp}(4, \mathbb{Z})$ matrix $M_{\text{dd}} = \begin{pmatrix} A_{\text{dd}} & B_{\text{dd}} \\ C_{\text{dd}} & D_{\text{dd}} \end{pmatrix}$ chosen such that the image of the diagonal divisor $\mathcal{D}_{\text{diag}} := \{\rho^2 = 0\}$ is essentially the dominant divisor given by $\mathcal{D}_{\text{dom}} := \{\rho^2 - \rho - \tau\sigma = 0\}$. The transformation behavior of the weight w Siegel form Φ_w then suggests the introduction of the form

$$\tilde{\Phi}_w(T) := \det(C_{\text{dd}}T' + D_{\text{dd}})^w \Phi_w(T'), \quad (38)$$

where the coordinates T' are defined as $T = M_{\text{dd}}T'$. With this form the degeneracies are defined by

$$d(Q_e, Q_m) = (-1)^{Q_e Q_m + 1} \int dT \frac{e^{-\pi i Q^t T Q}}{\tilde{\Phi}_w(T)}, \quad (39)$$

where $dT = d\tau d\sigma d\rho$, $Q = \begin{pmatrix} Q_m \\ Q_e \end{pmatrix}$, and Q_e, Q_m are the charges associated to the gauge fields of the CHL_N gauge fields. The microscopic entropy of black holes of charge (Q_e, Q_m) is then defined by

$$S^{\text{mic}}(Q_e, Q_m) = \ln |d(Q_e, Q_m)|. \quad (40)$$

This entropy has been shown to agree in certain approximations with the macroscopic entropy derived e.g. via the OSV framework, along the lines of ref. [16].

3 Black holes as probes of spacetime geometry

The idea of using black holes as probes of spacetime geometry motivates the question what precisely the information is that is encoded in their entropy. Specifically, we can ask whether we can learn something about the structure of the extra dimensions from the entropy of black holes. In this section we formulate the framework in which these vague questions can be made precise.

The microscopic entropy framework provided by the Siegel forms via (38), (39), (40) shows that the information encoded in the entropy is completely determined by the Siegel form Φ^N . Once this form has been identified, the degeneracies and the entropy are known. To determine the geometric origin of the black hole entropy thus means to determine the geometric origin of the Siegel modular forms.

In the context of the CHL_N models, viewed in the type II string framework as compactifications $\text{CHL}_N = \text{IIA}(X_N)$ on varieties of the type

$$X_N = (K3 \times E)/\mathbb{Z}_N,$$

it is natural to ask whether the modular forms that arise in the black hole entropy functions can be used as probes that would allow to deduce information about the compact geometry if one could perform experiments with CHL_N black holes in the laboratory. With a physical probe that is highly sensitive to the details of the ambient spacetime one might hope to be able to reconstruct completely the precise structure of this geometry. We will show that for the entropy of CHL_N black holes this is not the case. The relevant geometric information that is encoded in the entropic Siegel modular forms is motivic (see below) in the sense that a single geometric structure suffices to determine these forms completely. In general a variety, in particular varieties of the form X_N describing the compactification of the CHL_N models, lead to several motivic building blocks that characterize the manifold completely, while a single motive can often be embedded into different varieties.

3.1 Automorphic motives as geometric building blocks

The problem of finding the geometric origin of automorphic forms leads to the notion of motives, certain substructures of manifolds that are reflected in the cohomology of a variety. Manifolds are not to be viewed as single monolithic objects but instead as a coherent structure of several different building blocks, where the same building blocks can appear in different spaces. Intuitively motives behave therefore like fundamental particles. A brief physics oriented discussion of motives and some of their applications can be found in [6], and an extensive mathematical treatment is contained in [12].

Over the past hundred years it has become clear through much mathematical experimentation that motives M often are automorphic in the sense that their L-functions $L(M, s)$, viewed as complex functions, are identical to L-functions associated to automorphic forms $L(\Phi, s)$

$$L(M, s) = L(\Phi, s).$$

The only rigorously established case is the geometric interpretation of modular forms of weight two with respect to congruence groups of level N in terms of motives associated to elliptic curves of conductor N . First steps in this direction were taken by Klein and Hurwitz in the late 19th century, but a general result was proven only a century later by Wiles and Taylor for stable elliptic curves [13], and in complete generality in ref. [14]. For higher dimensional motives the automorphic framework is more involved, and no general results are known. However, a number of conjectures about the expected structure of automorphic forms and representations have been formulated within the context of the Langlands program. This allows us to compare the conjectured motivic structure of Siegel modular forms with the compactification geometry of the CHL_N model.

3.2 Genus n Siegel modular motives

The fact that the partition function of black holes in CHL_N models is determined completely by genus two Siegel modular forms Φ^N implies that the problem of finding a geometric origin of the entropy of CHL_N black holes translates into the question whether it is possible to reverse engineer for each of the forms Φ^N one or several modular motives M_N such that the resulting motivic modular forms lead to the Siegel forms in a canonical way. The general philosophy of the Langlands program suggests that this question makes sense in the context of any physical automorphic form, but in general this is an unsolved problem. The web of conjectures formulated by Langlands and others makes it possible however to make some general observations about the geometric interpretation of Siegel modular forms without specifying the degree of supersymmetry, or the type of compactification associated to it.

While the theory of automorphic forms is not developed far enough to allow universal statements even at the conjectural level, for Siegel modular forms of arbitrary genus n the expected motives can be characterized in a way that is precise enough to evaluate their physical relevance. Two quantities that give numerical information about the ambient variety are the weight $\text{wt}(M)$ and the rank $\text{rk}(M)$ of the motive M . The former determines the cohomology group $H^i(X)$ in which the cohomological realization $H(M)$ of the motive M lives. For Siegel

forms the natural generalization of classical modular motives is the spinor motive M_Φ , which is characterized by its particular form of the L-function in terms of the Satake parameters. For forms of weight w and genus n the weight $\text{wt}(M_\Phi)$ of the induced spinor motive M_Φ can be read off the expected functional equation to be given by

$$\text{wt}(M_\Phi) = nw - \frac{n}{2}(n+1), \quad (41)$$

while the rank of the Siegel modular motive can be read off the spinor L-function as given by

$$\text{rk}(M_\Phi) = 2^n. \quad (42)$$

Furthermore, the Hodge decomposition of the cohomological realization $H(M_\Phi)$ of M_Φ is given for genus two forms by

$$H(M_\Phi) = H^{2w-3,0} \oplus H^{w-1,w-2} \oplus H^{w-2,w-1} \oplus H^{0,2w-3}. \quad (43)$$

These results show that even though for genus two Siegel forms the rank of the motive is realistic within the framework of Calabi-Yau threefold compactifications, the weight of genus two Siegel modular motives leads to varieties of dimension $\dim_{\mathbb{C}} X = 2w - 3$. For several of the CHL_N models the dimension is therefore too high.

We have seen above that the CHL_N Siegel forms are determined via a canonical construction in terms of Maaß-Skoruppa roots f^N that are classical modular forms. It is therefore natural to ask whether these forms lead to modular motives that can be accommodated by string compactifications.

3.3 Motives of classical modular forms

The fact that the Siegel forms Φ^N are determined completely in terms of classical modular forms f^N via the Maaß-Skoruppa lift shows that the essential information pertaining to the different models for varying N is completely contained in the forms f^N . Asking for a geometric origin of the Φ^N therefore mean to find a geometric interpretation of f^N .

Classical modular forms $f \in S_w(\Gamma_0(N), \epsilon_N)$ can be viewed as genus one forms, leading to motives M_f of weight

$$\text{wt}(M_f) = w - 1 \quad (44)$$

and rank $\text{rk}(M_f) = 2$, with Hodge structure given by

$$H(M_f) = H^{w-1,0} \oplus H^{0,w-1}. \quad (45)$$

In the case of the CHL_N models compactified on X_N the question becomes how one can determine motives M_N as a combination of $K3, E, N$. We therefore can ask the more concrete question whether the modular forms involved in the black hole entropy are determined by motives of the type

$$M_N = M(K3, E, N) \quad (46)$$

possibly including the precise form of the action of the group \mathbb{Z}_N .

The Maaß-Skoruppa roots f^N , describing the purely electrically charged states of the CHL_N models, do not appear to follow any obvious pattern that would suggest a geometric origin. The range of modular weights is between three and twelve, which makes a direct spacetime interpretation via Calabi-Yau varieties again impossible for similar reasons that Siegel modular motives cannot be physical for the CHL_N black holes. While the Hodge structure $H^{w-1,0} \oplus H^{0,w-1}$ does exist in any Calabi-Yau variety, the dimension of the variety supporting the motive is too high for the CHL_N models because the cohomological realization $H(M_\Omega)$ of the motive is defined by a Galois orbit in the intermediate cohomology $H(M_\Omega) \subset H^n(X_n)$, where $n = \dim_{\mathbb{C}} X_n$. If the motive is pure the weight w of the corresponding modular form f_M (if it exists) is given by the complex dimension as $w(f_M) = n + 1$. For the CHL_N models this implies that a direct Calabi-Yau interpretation of the Maaß-Skoruppa root would have to involve manifolds of complex dimensions ranging from two to eleven. As a result most of the dimensions that appear in the CHL_N models are too high to be derived from a compactification space within string theory, M-theory, or F-theory.

What turns the black hole entropy of CHL_N into a probe that is manageable is the fact that the CHL_N Siegel modular forms can be built from modular forms of weight two, thereby

extending the Maaß-Skoruppa lift one step further, as we will show in the next section. Our constructions work by adding to the lift diagram (20) one further reduction $f_2^{\tilde{N}} \longrightarrow f^N$.

4 The motivic origin of CH_N black hole entropy

The fact that neither the Siegel forms Φ^N nor their Maaß-Skoruppa roots f^N lead to physical motives raises the question whether the Maaß-Skoruppa lift can be pushed further, i.e. whether the forms f^N can in turn be constructed in some way from even simpler building blocks, and if so, whether those building blocks admit a geometric interpretation. Given the structure of the extra dimensions in the CHL_N models it would be natural to expect that the ingredients in such a construction might involve modular forms of weight two, associated to elliptic curves, or modular forms of weight three, associated to K3 surfaces, or both. It is this problem that we address in this section.

It turns out that the sequence of classical black hole forms given by the Maaß-Skoruppa roots splits into two distinct and disjoint classes of forms, hence no completely universal reduction should be expected. The fact that these classes form disjoint sets guarantees that the elliptic reductions we describe are unique (up to isogeny). The property that distinguishes certain of the forms f^N is concerned with their symmetry structure. While the majority of the black hole forms $f^N \in S_{w+2}(\Gamma_0(N), \epsilon_N)$ have no particular symmetry, the forms at levels $N = 4, 7, 8$ admit a particularly simple structure because they are of complex multiplication (CM) type. We will answer the question raised above about the geometric origin of the CHL_N entropy by showing that all CHL_N Maaß-Skoruppa roots f^N can be constructed in terms of classical modular forms of weight two $f_2^{\tilde{N}}(q) \in S_2(\Gamma_0(\tilde{N}))$ via two different reductions, one for the non-CM type forms, the other for the CM type forms. These modular forms are supported by the unique motives of elliptic curves $E_{\tilde{N}}$ of conductor \tilde{N} that are determined up to isogeny. We describe these two reductions in the following subsections.

4.1 Non-CM elliptic reduction of CHL_N Maaß-Skoruppa roots

Our first reduction applies to those CHL_N models for which the black hole root f^N has no CM, i.e. $N = 1, 2, 3, 5, 6$. The key observation here is that for these models we write the higher weight classical black hole modular forms $f^N(q)$ defining the Maaß-Skoruppa roots in terms of classical modular forms of weight two as follows. For each order N as given, there exists an integer \tilde{N} and a cusp form of level \tilde{N} and weight two, $f_2^{\tilde{N}}(q) \in S_2(\Gamma_0(\tilde{N}))$, such that the classical black hole form $f^N \in S_{w+2}(\Gamma_0(N), \epsilon_N)$ can be written as

$$f^N(q) = f_2^{\tilde{N}}(q^{1/m})^m \quad (47)$$

where

$$m = \frac{1}{2} \left\lceil \frac{24}{N+1} \right\rceil. \quad (48)$$

The relation between the order N of the CHL_N group \mathbb{Z}_N and the level \tilde{N} of the weight two form is given in Table 2.

| Order N | 1 | 2 | 3 | 5 | 6 |
|-------------------|----|----|----|----|----|
| Level \tilde{N} | 36 | 32 | 27 | 20 | 24 |

Table 2. *The levels \tilde{N} in terms of the orders N of the CHL_N models.*

The construction (47) of the Maaß-Skoruppa root from building blocks given by classical modular forms of weight two then leads to a geometric interpretation in terms of elliptic curves by applying the elliptic modularity theorem proven by Wiles, Taylor and others [13]. This theorem proves the conjecture Taniyama-Shimura-Weil conjecture, according to which every elliptic curve over the rational numbers is modular in the sense that its L-function agrees with the L-function of a weight two form. Weil in particular made this somewhat vague conjecture more precise by his important experimental observation that for an elliptic curve E_N of conductor N the associated modular form of weight 2 is associated to a Hecke congruence group $\Gamma_0(N)$ at level N . Given our weight two modular forms $f_2^{\tilde{N}}(q) \in S_2(\Gamma_0(\tilde{N}))$ derived from the black hole forms $f^N(q)$ we can therefore construct elliptic curves $E_{\tilde{N}}$ of conductor \tilde{N} such that their associated modular forms $f_2(E_{\tilde{N}}, q)$ are given by the elliptic

roots

$$f_2(E_{\tilde{N}}, q) = f_2^{\tilde{N}}(q). \quad (49)$$

We will call the forms $f_2^{\tilde{N}}(q)$ the elliptic roots of the Siegel forms Φ^N . The proof of this relation can be given explicitly via a case by case analysis for $N = 1, 2, 3, 5, 6$, without the abstract machinery that enters Wiles' proof of the Taniyama-Shimura-Weil conjecture for stable elliptic curves, and the more general proof for all elliptic curves by Breuil, Conrad, Diamond and Taylor [13].

The reduction (47) turns out to lead to elliptic curves $E_{\tilde{N}}$ which admit complex multiplication for $N = 1, 2, 3$, and to curves with no CM for $N = 5, 6$.

This leaves the CHL_N models with $N = 4, 7, 8$. For these cases the reduction (47) cannot be applied because the Maaß-Skoruppa roots f^N have odd weight. For this class of forms it is necessary to introduce a different type of reduction, to which we turn in the next subsection.

4.2 CM type elliptic reduction

As mentioned above, the Maaß-Skoruppa roots f^N for the CHL_N models fall into two different classes of different symmetry types. While for $N = 1, 2, 3, 5, 6$ these forms do not exhibit any particular symmetries, for $N = 4, 7, 8$ these forms admit complex multiplication. As a consequence, these forms are sparse in the sense that their coefficients a_p vanish for half the primes. It is this property which we will use to complete our elliptic reduction for the remaining CHL_N models.

There are several ways to think about complex multiplication forms $f(\tau) \in S_w(\Gamma_0(N), \epsilon)$. The point of view that explains the vanishing behavior of its Fourier coefficients in the most direct way is encoded in the definition originally given by Ribet [15]. The key here is that associated to each CM form is an imaginary quadratic field $K_D = \mathbb{Q}(\sqrt{-D})$, with D square free, such that the coefficients a_p , for p prime, of its Fourier series $f(q) = \sum_n a_n q^n$, vanish for precisely those rational primes p that do not split in the ring of integers \mathcal{O}_K of the field K_D . The splitting behavior of the rational primes within \mathcal{O}_{K_D} is controlled by the Legendre symbol χ_D :

if $\chi_D(p) = 1$ then the prime p factors in \mathcal{O}_{K_D} . A CM form $f \in S_w(\Gamma_0(N), \epsilon)$ therefore can be defined through its expansion by the condition that there exists a field K_D such that

$$\chi_D(p)a_p = a_p. \quad (50)$$

To make the elliptic reduction of the black hole forms f^N for the CHL_N models with $N = 4, 7, 8$ more transparent it is useful to shift perspective, and to consider the L -functions associated to the modular forms f^N , defined by the Mellin transform. Given the Fourier expansion $f(q) = \sum_n a_n q^n$ of any cusp form $f \in S_w(\Gamma_0(N), \epsilon)$, the Mellin transform associates to f the L -series

$$L(f, s) = \sum_n \frac{a_n(f)}{n^s}. \quad (51)$$

The fact that the modular forms $f^N(q)$ for the CHL_N models with $N = 4, 7, 8$ have complex multiplication means that their L -functions can be identified with the L -series of algebraic Hecke characters Ψ_N associated to extensions K_N of the rational field \mathbb{Q} . In the present discussion the relevant fields are imaginary quadratic extensions $K_N = \mathbb{Q}(\sqrt{-D_N})$, where D_N is a square free integer.

Algebraic Hecke characters associated to imaginary quadratic fields are defined by a congruence ideal $\mathfrak{m} \subset \mathcal{O}_K$. If $\mathcal{I}_{\mathfrak{m}}$ denotes the fractional ideals prime to \mathfrak{m} , algebraic Hecke characters are maps

$$\Psi : \mathcal{I}_{\mathfrak{m}} \longrightarrow \mathbb{C}^\times \quad (52)$$

that can be normalized to be given on the principal ideals as

$$\Psi((z)) = z^w, \quad \text{with } z \equiv 1 \pmod{\times \mathfrak{m}}, \quad (53)$$

where mod^\times denotes multiplicative congruence. The integer w denotes the weight of Ψ .

Given an algebraic Hecke character Ψ of conductor \mathfrak{c}_Ψ , associated to an imaginary field K of discriminant $-D$, define integers a_n by summing over all integral ideals \mathfrak{a} of K coprime to \mathfrak{c}_Ψ as

$$a_n = \sum_{\substack{(\mathfrak{a}, \mathfrak{c}_\Psi)=1 \\ N\mathfrak{a}=n}} \Psi(\mathfrak{a}), \quad (54)$$

where $N\mathfrak{a}$ denotes the norm of the ideal \mathfrak{a} . The L -series of the character Ψ is defined as

$$L(\Psi, s) = \sum_{n=1}^{\infty} \frac{a_n(\Psi)}{n^s}. \quad (55)$$

It is a theorem of Hecke that the q -series $f(\Psi, q) = \sum_n a_n(\Psi)q^n$, with $q = e^{2\pi i\tau}$, associated to $L(\Psi, s)$ via the inverse Mellin transform defines a modular form of weight $(w+1)$, level

$$N_\Psi = DN\mathfrak{c}_\Psi, \quad (56)$$

and its Nebentypus character given by

$$\epsilon_\Psi(m) = \frac{1}{m^w} \chi_D(m) \Psi((m)). \quad (57)$$

Here $\chi_D(m)$ is the Legendre symbol defined above. i.e. $f(\Psi, q) \in S_{w+1}(\Gamma_0(N_\Psi), \epsilon_\Psi)$.

For the CHL_N models with groups of orders $N = 4, 7, 8$, the imaginary quadratic fields $K_N = \mathbb{Q}(\sqrt{-D_N})$ are given by the Gauss field $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-7})$, $\mathbb{Q}(\sqrt{-2})$, respectively. These fields are all of class number one, hence all the ideals are principal.

The CM nature of the modular forms $f^N \in S_{w+2}(\Gamma_0(N), \epsilon_N)$ implies that the essential information of these forms is encoded in modular forms of weight 2 at levels \tilde{N} that depend on the order N of the quotient groups \mathbb{Z}_N . This can be seen as follows. First, there exist algebraic Hecke characters $\Psi_{\tilde{N}}$ such that the L -series of the black hole forms are given by powers $\Psi_{\tilde{N}}^{w+1}$ of the characters $\Psi_{\tilde{N}}$

$$L(f^N, s) = L(\Psi_{\tilde{N}}^{w+1}, s), \quad (58)$$

for $(N, \tilde{N}) = (4, 32), (7, 49), (8, 256)$. The elliptic origin of the classical black hole forms f^N now follows from the fact that the L -function $L(\Psi_{\tilde{N}}, s)$ of the Hecke character $\Psi_{\tilde{N}}$ is the Mellin transforms of a weight two modular form $f_2^{\tilde{N}} \in S_2(\Gamma_0(\tilde{N}))$. It can furthermore be shown that the Mellin transform of these forms $f_2^{\tilde{N}}$ agree with the L -series of elliptic curves $E_{\tilde{N}}$ with conductor \tilde{N} which admit complex multiplication by K_D

$$L(E_{\tilde{N}}, s) = L(\Psi_{\tilde{N}}, s) = L(f_2^{\tilde{N}}, s). \quad (59)$$

In this way we obtain a second systematic construction that leads to an elliptic curve interpretation for the remaining CHL_N models, thereby completing our elliptic reduction of the Siegel forms Φ^N for all CHL_N models.

The Hecke L -series interpretation (58) of the black hole forms f^N for $N = 4, 7, 8$, combined with (59), gives the most systematic formulation of the link between the high weight forms f^N and the weight two forms $f_2^{\tilde{N}}$. For the examples of the present paper it is possible to express the coefficients of the black hole forms f^N in a more direct, but less transparent, way in terms of the coefficients of the classical weight two forms $f_2^{\tilde{N}}$. Expanding the forms as

$$f^N(q) = \sum_n b_n^{w+2} q^n, \quad f_2^{\tilde{N}}(q) = \sum_n a_n q^n \quad (60)$$

one can derive for the coefficients at primes p the relations

$$\begin{aligned} b_p^3 &= a_p^2 - 2p \\ b_p^5 &= a_p^4 - 4pa_p^2 + 2p^2. \end{aligned} \quad (61)$$

The relevant elliptic forms of weight two for $N = 4, 7, 8$ are given in Table 3. For $N = 7, 8$ we use the symbol \cong to indicate that we are listing only the coefficients a_p for primes p . These are the important coefficients because all other Fourier coefficients can be obtained via the Hecke relations

$$\begin{aligned} a_{p^n} &= a_p a_{p^{n-1}} + p^w a_{p^{n-2}} \\ a_{mn} &= a_m a_n, \quad \text{for } m \neq n, \end{aligned} \quad (62)$$

because the forms of weight two are Hecke eigenforms. Combining these results with those of Table 2 gives the conductors \tilde{N} of the elliptic curve $E_{\tilde{N}}$ associated to each of the CHL_N model associated to a quotient manifold with respect to \mathbb{Z}_N .

| $(w+2, N, \tilde{N})$ | K | $f_2^{\tilde{N}} \in S_2(\Gamma_0(\tilde{N}))$ |
|-----------------------|-------------------------|--|
| $(5, 4, 32)$ | $\mathbb{Q}(\sqrt{-1})$ | $f_2^{32}(q) = \eta(q^4)^2 \eta(q^8)^2$ |
| $(3, 7, 49)$ | $\mathbb{Q}(\sqrt{-7})$ | $f_2^{49}(q) \cong q + q^2 + 4q^{11} + 8q^{23} + 2q^{29} - 6q^{37} - 12q^{43} + \dots$ |
| $(3, 8, 256)$ | $\mathbb{Q}(\sqrt{-2})$ | $f_2^{256}(q) \cong q + 2q^3 + 6q^{11} - 6q^{17} + 2q^{19} + 6q^{41} - 10q^{43} + \dots$ |

Table 3. *Elliptic forms associated to the CHL_N models for $N = 4, 7, 8$.*

By considering the expansion of the congruence group black hole forms f^N for $N = 4, 7, 8$ it can be checked that the coefficients their L -functions are given in terms of the relations (61). We will call this construction of the high weight Maaß-Skoruppa roots f^N in terms of classical modular forms of weight two the CM-type elliptic reduction.

4.3 Elliptic weight two roots of the CHL_N Maaß-Skoruppa roots

As a result of our two lift constructions (47) and (58), we have established that the geometric origin of the black hole forms $\Phi^N \in S_w(\Gamma_0^{(2)}(N))$ is mediated by the interpretation of the classical forms $f^N \in S_{w+2}(\Gamma_0(N), \epsilon_N)$ in terms of weight two forms $f_2^{\tilde{N}} \in S_2(\Gamma_0(\tilde{N}))$. The elliptic curves $E_{\tilde{N}}$ whose motives support the modular forms $f_2^{\tilde{N}}$ are determined up to isogeny, i.e. maps that are surjective and have a finite kernel. The results of this geometric interpretation are summarized in Table 4, which also includes an indication whether the forms that appear are of complex multiplication type or not.

| N of CHL_N | BH Form $f^N(q) \in S_{w+2}(\Gamma_0(N))$ | Motivic form $f_2^{\tilde{N}}(q)$ | Level \tilde{N} of $E_{\tilde{N}}$ |
|--------------------------|---|--|---|
| 1 | $\eta(\tau)^{24}$ | CM $\eta(q^6)^4 \in S_2(\Gamma_0(36))$ | 16 |
| 2 | $\eta(\tau)^8 \eta(2\tau)^8$ | CM $\eta(q^4)^2 \eta(q^8)^2 \in S_2(\Gamma_0(32))$ | 32 |
| 3 | $\eta(\tau)^6 \eta(3\tau)^6$ | CM $\eta(q^3)^2 \eta(q^9)^2$ | 27 |
| 4 | CM $\eta(\tau)^4 \eta(2\tau)^2 \eta(4\tau)^4$ | $\text{Sym}^4(f_2^{32})$ with $f_2^{32} \in S_2(\Gamma_0(32))$ | 32 |
| 5 | $\eta(\tau)^4 \eta(5\tau)^4$ | $\eta(q^2)^2 \eta(q^{10})^2 \in S_2(\Gamma_0(20))$ | 20 |
| 6 | $(\eta(\tau) \eta(2\tau) \eta(3\tau) \eta(6\tau))^2$ | $\eta(2\tau) \eta(4\tau) \eta(6\tau) \eta(12\tau) \in S_2(\Gamma_0(24))$ | 24 |
| 7 | CM $\eta(\tau)^3 \eta(7\tau)^3$ | $\text{Sym}^2 f_2^{49}$ with $f_2^{49} \in S_2(\Gamma_0(49))$ | 49 |
| 8 | CM $\eta(\tau)^2 \eta(2\tau) \eta(4\tau) \eta(8\tau)^2$ | $\text{Sym}^2 f_2^{256}$ with $f_2^{256} \in S_2(\Gamma_0(256))$ | 256 |

Table 4. *Motives associated to the electric modular forms of CHL_N models.*

We see from the reductions compiled in Table 4 that the elliptic lifts $f_2^{\tilde{N}} \rightarrow f^N$ of the CHL_N modular forms lead to classical modular forms of weight two that admit CM for $N = 1, 2, 3$. Combining this with the three Maaß-Skoruppa roots f^N at $N = 4, 7, 8$ leaves the two forms at $N = 5, 6$ that do not have CM. This indicates that CM is not a fundamental property as far as the geometric structure of these models is concerned. We show in the Appendix that it is possible to construct these two non-CM forms in terms of non-geometric forms of weight 1 that do admit complex multiplication.

5 Conclusions

In this paper we have described a program to view black holes as probes of the geometry of extra dimensions – given a hypothetical black hole in the laboratory, we can ask what the data extracted from this black hole might tell us about the details of the small scale structure of spacetime. We have shown that this strategy can be made concrete in the context of automorphic black holes by combining it with the idea that automorphic forms are supported by motives. While the precise framework of such automorphic motives is not known at present, certain concrete features are expected to be present for motives that support such forms.

Assuming that the conjectured properties of automorphic motives we see that in the context of Siegel automorphic black hole entropy the motives induced by the Siegel forms are not physical for dimensional reasons. Nor are the motives induced by the Maaß-Skoruppa roots which count $\frac{1}{2}$ -BPS states for the same reason. The key to the geometry is the realization that the Maaß-Skoruppa roots that appear in the class of CHL_N models are in fact of a special type such that they can be constructed from modular cusp form of weight two. We have shown that independent of the complex multiplication properties of the Maaß-Skoruppa root it is possible to induce the black hole roots f^N from forms $f_2 \in S_2(\Gamma_0(\tilde{N}))$ with $\tilde{N}(N)$, leading to a lift diagram that extends the usual Maaß-Skoruppa lift

$$f_2^{\tilde{N}} \longrightarrow f^N \xrightarrow{\text{Sk}} \varphi^N \xrightarrow{\text{MS}} \Phi^N,$$

where we have reinstated the weight $(w+2)$ given by (36). These forms of weight two in turn are supported by elliptic curves.

The fact that the Siegel forms Φ^N are induced by elliptic motives shows that the Siegel count of the CHL_N black hole entropy only contains a limited amount of information about the geometry of spacetime. It would be of interest to refine the black hole formulae so as to encode more detailed motivic information that allows to reconstruct spacetime more precisely. A possible strategy in this direction would be to consider more general black holes [17] than have been considered so far for the class of CHL_N models.

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6 Appendix: CM properties of black hole entropy

The list of forms in Table 4 shows that three out of the 8 CHL_N Maaß-Skoruppa roots that describe the small black holes in these models are of CM type, namely those at $N = 4, 7, 8$. The elliptic forms of weight two obtained for the remaining classical forms via the non-CM type reduction (47) are of CM type for $N = 1, 2, 3$, leading to elliptic curves with complex multiplication symmetry. This leaves two forms and their associated elliptic curves without CM, and raises the question there these non-CM type modular forms cannot be built in some other way from forms that do admit CM, if perhaps in a non-geometric way. It turns out that the answer is affirmative.

The remaining two forms at $N = 5, 6$ can be constructed in terms of CM forms of weight one as

$$\begin{aligned} f_{1,80}(\tau) &= \eta(4\tau)\eta(20\tau) \in S_1(\Gamma_1(80)) \\ f_{1,128}(\tau) &= \eta(8\tau)\eta(16\tau) \in S_1(\Gamma_1(128)). \end{aligned} \tag{63}$$

With these forms we can write

$$\begin{aligned} f_2^{20}(q) &= f_1^{80}(q^{1/2})^2 \\ f_2^{24}(q) &= f_1^{128}(q^{1/4})f_{1,128}(q^{3/4}). \end{aligned} \tag{64}$$

This shows that if one considers reductions to non-geometric modular forms of weight 1 all the black hole modular forms can be constructed in terms of CM modular forms.

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